

## Lecture 6

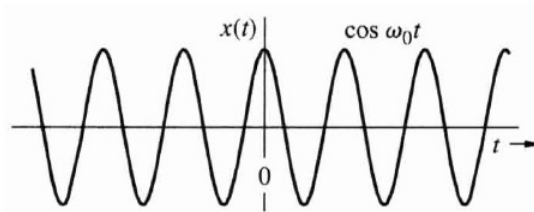
# Windowing Effects & Discrete Fourier Transform

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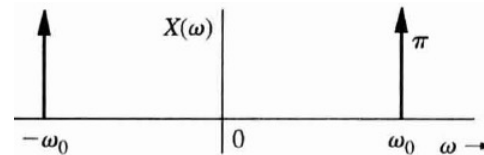
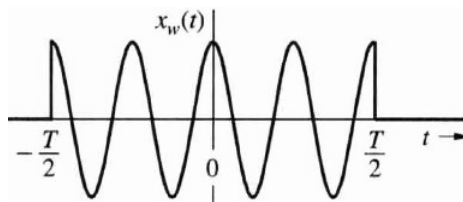
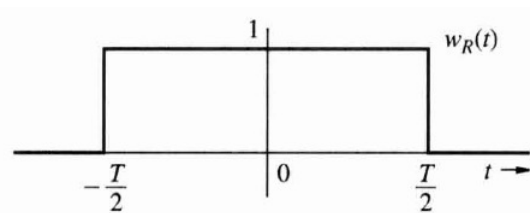
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# Windowing and its effect

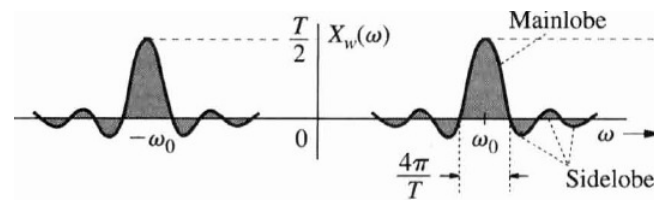
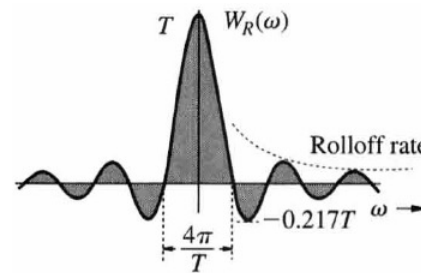
- ◆ Extracting a segment of a signal in time is the same as multiplying the signal with a rectangular window:



**X**



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## Spectral spreading

Energy spread out from  $\omega_0$  to width of  $2\pi/T$  – reduced spectral resolution.

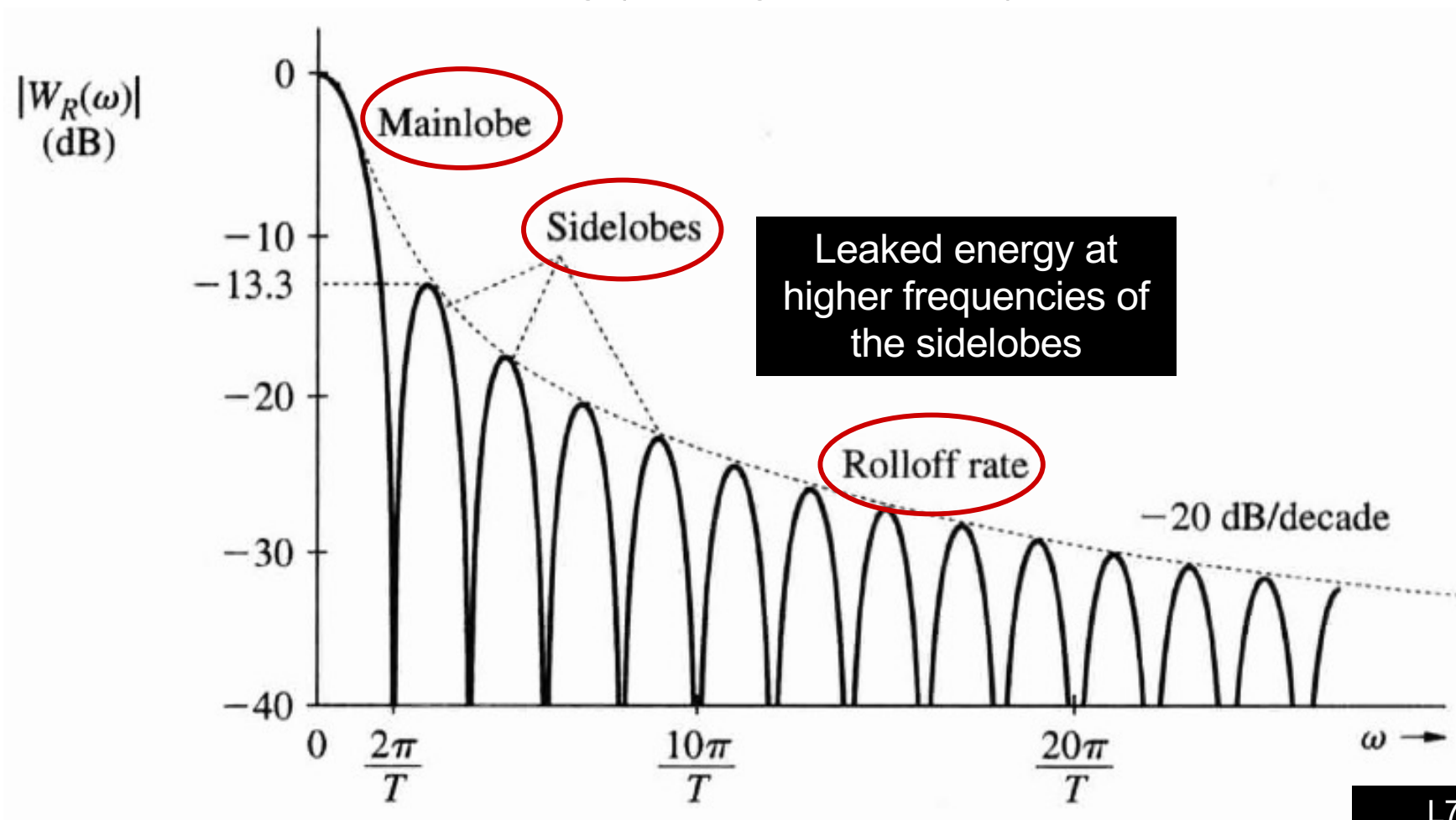
## Leakage

Energy leaks out from the mainlobe to the sidelobes.

**L7.8**

# Mainlobe & Sidelobes in dB

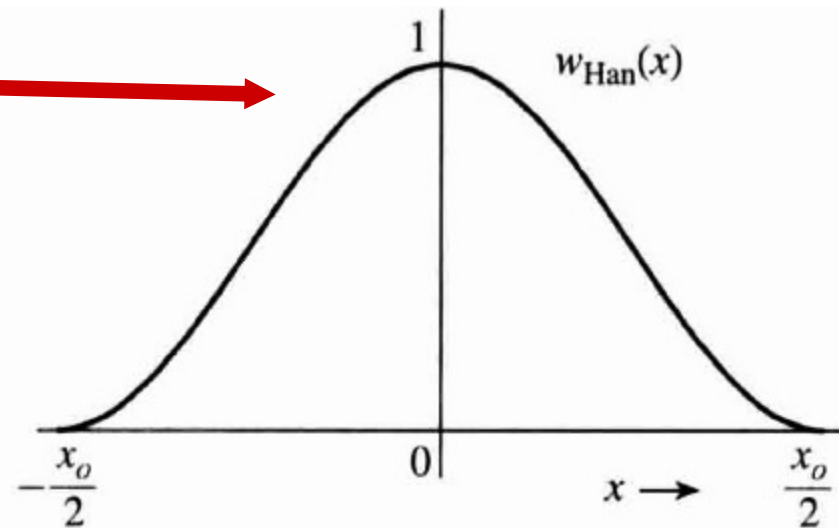
- ◆ Detail effects of windowing (rectangular window):



L7.8

# Remedies for side effects of truncation

1. Make mainlobe width as narrow as possible → implies as wide a window as possible.
  2. Avoid big discontinuity in the windowing function to reduce leakage (i.e. high frequency sidelobes).
  3. 1) and 2) above are incompatible – therefore needs a compromise.
- ◆ Commonly replace rectangular window with one of these:
- Hamming window
  - Hanning window
  - Barlett window
  - Blackman window
  - Kaiser window



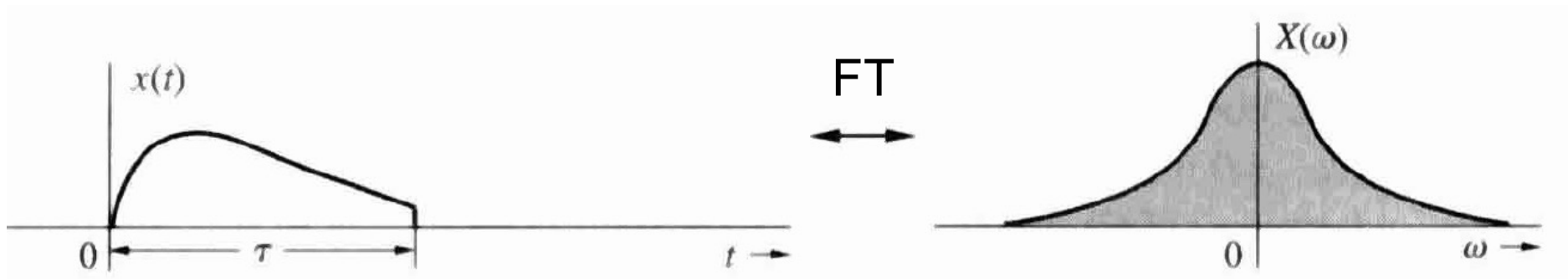
L7.8

# Comparison of different windowing functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5
3	Hanning: $0.5 \left[ 1 + \cos\left(\frac{2\pi t}{T}\right) \right]$	$\frac{8\pi}{T}$	-18	-31.5
4	Hamming: $0.54 + 0.46 \cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7
5	Blackman: $0.42 + 0.5 \cos\left(\frac{2\pi t}{T}\right) + 0.08 \cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1
6	Kaiser: $\frac{I_0 \left[ \alpha \sqrt{1 - 4 \left(\frac{t}{T}\right)^2} \right]}{I_0(\alpha)}$ $0 \leq \alpha \leq 10$	$\frac{11.2\pi}{T}$	-6	-59.9 ( $\alpha = 8.168$ )

# Spectral Sampling (1)

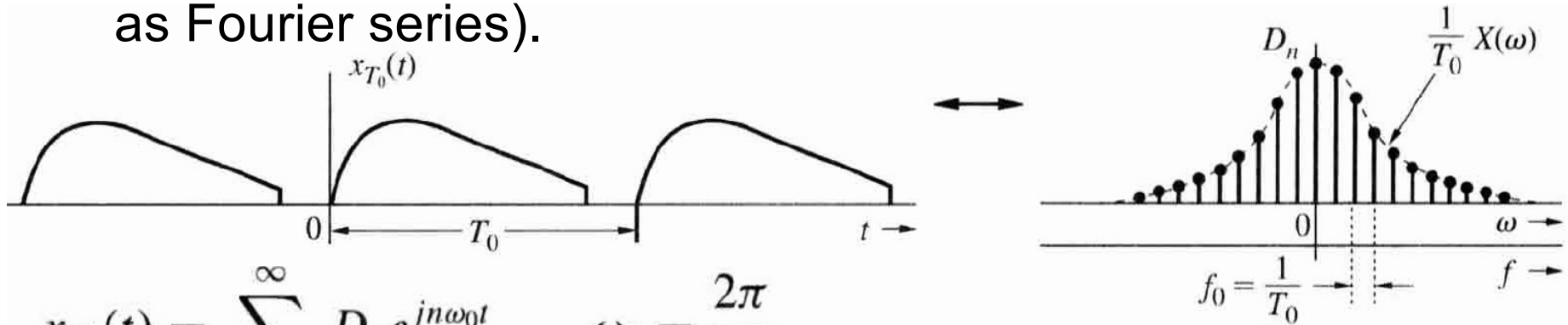
- ◆ As expected, time domain sampling has a dual: spectral sampling.
- ◆ Consider a time limited signal  $x(t)$  with a spectrum  $X(\omega)$ .



$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_0^{\tau} x(t)e^{-j\omega t} dt$$

## Spectral Sampling (2)

- ◆ If we now CONSTRUCT a periodic signal  $x_{T_0}(t)$ , we will expect the spectrum of this signal to be discrete (expressed as Fourier series).



$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \omega_0 = \frac{2\pi}{T_0}$$

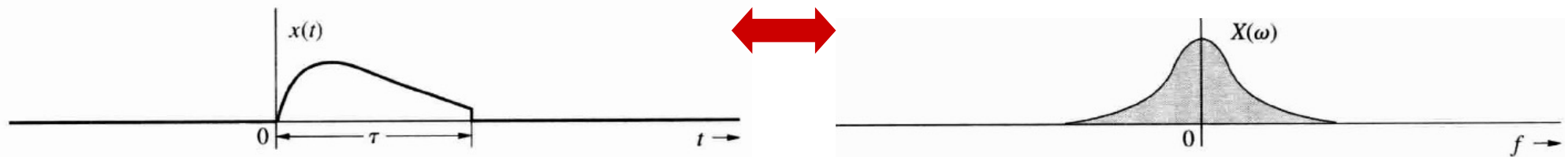
where  $D_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_0^{\tau} x(t) e^{-jn\omega_0 t} dt$

therefore

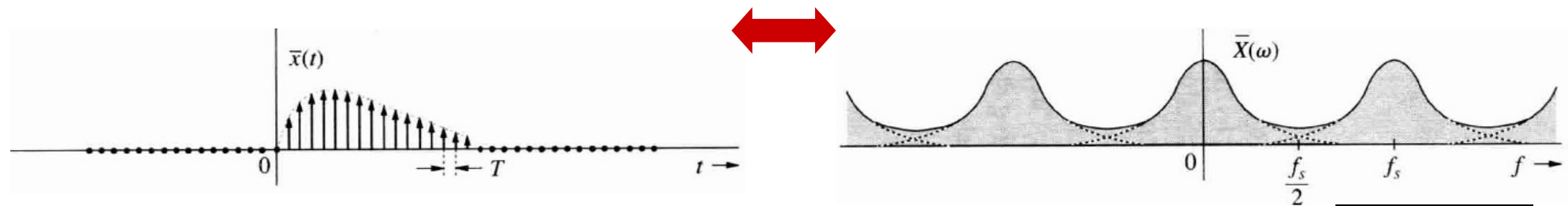
$$D_n = \frac{1}{T_0} X(n\omega_0)$$

# The Discrete Fourier Transform (DFT) (1)

- ◆ Fourier transform is computed (on computers) using discrete techniques.
- ◆ Such numerical computation of the Fourier transform is known as Discrete Fourier Transform (DFT).
- ◆ Begin with time-limited signal  $x(t)$ , we want to compute its Fourier Transform  $X(\omega)$ .

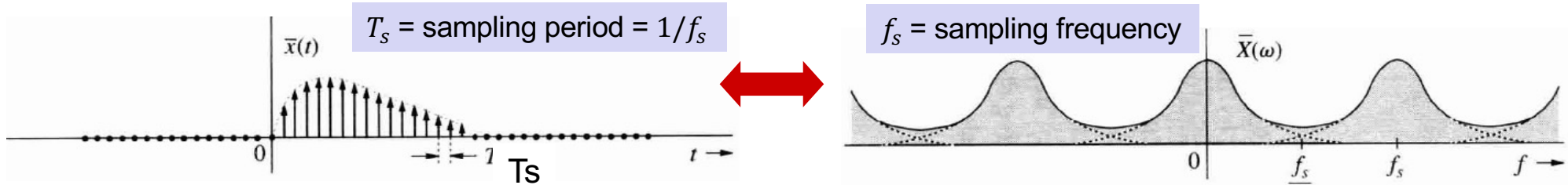


- ◆ We know the effect of sampling in time domain:

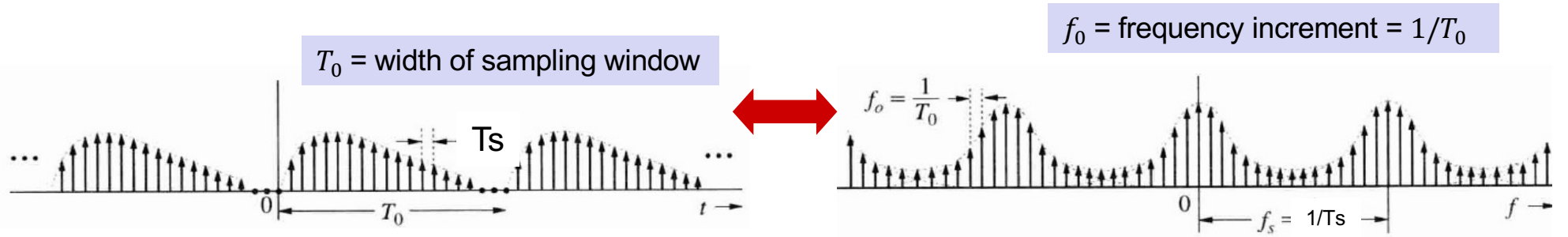


L8.5

# The Discrete Fourier Transform (DFT) (2)



- ◆ Now construct the sampled version of  $x(t)$  as repeated copies. The effect (from slides 6-8) is sampling the spectrum.



Number of time samples in  $T_0$

$$N_0 = \frac{T_0}{T_s}$$

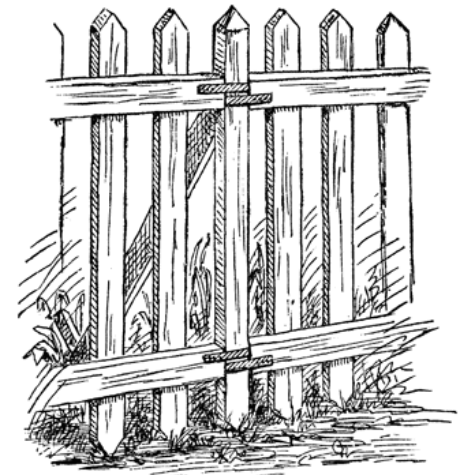
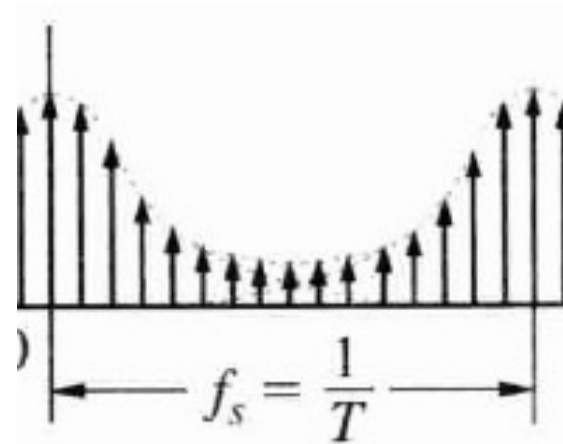
Number of frequency samples in  $f_s$

$$N'_0 = \frac{f_s}{f_0}$$

$$N_0 = \frac{T_0}{T_s} = \frac{1/f_0}{1/f_s} = \frac{f_s}{f_0} = N'_0$$

# Picket Fence Effect

- ◆ Numerical computation method yields uniform sampling values of  $X(\omega)$ .
- ◆ Information between samples in spectrum is missing – picket fence effect:
- ◆ Can improve spectral resolution by increasing number of samples used in the window  $N_0$ , i.e. the period of signal being transformed  $T_0$ .



## Formal definition of DFT

- ◆ If  $x[nT]$  and  $X[r\omega_0]$  are the  $n^{\text{th}}$  and  $r^{\text{th}}$  samples of  $x(t)$  and  $X(\omega)$  respectively, then we define:

$$x_n = T_s \times x[nT_s] = \frac{T_0}{N_0} x[nT] \quad \text{and} \quad X_r = X(r\omega_0)$$

where

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- ◆ Then

Forward DFT

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}$$

$$\Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

Inverse DFT

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

L8.5

# Parseval's Theorem

- The energy of a signal  $x(t)$  can be derived in time or frequency domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

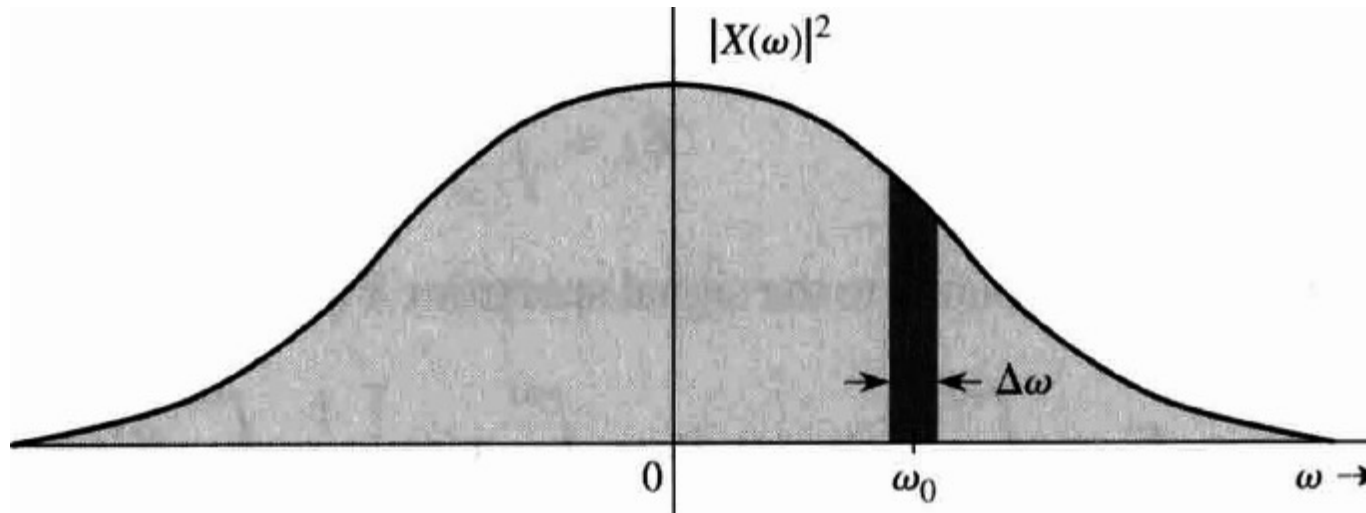
- Proof:

$$\begin{aligned}
 E_x &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t) dt && \boxed{x^*(t) \iff X^*(-\omega)} \\
 &= \int_{-\infty}^{\infty} x(t) \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega) d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega
 \end{aligned}$$

Change order of integration

# Energy Spectral Density of a signal

- ◆ Total energy is area under the curve of  $|X(\omega)|^2$  vs  $\omega$  (divided by  $2\pi$ ).



- ◆ The energy over a small frequency band  $\Delta\omega$  ( $\Delta\omega \rightarrow 0$ ) is:

$$\Delta E_x = \frac{1}{2\pi} |X(\omega)|^2 \Delta\omega = |X(\omega)|^2 \Delta f \quad \frac{\Delta\omega}{2\pi} = \Delta f \text{ Hz}$$

Energy spectral density (per unit bandwidth in Hz)

L7.6

# Energy Spectral Density of a REAL signal

- ◆ If  $x(t)$  is a real signal, then  $X(\omega)$  and  $X(-\omega)$  are conjugate:

$$|X(\omega)|^2 = X(\omega)X^*(\omega) = X(\omega)X(-\omega)$$

- ◆ This implies that  $X(\omega)$  is an even function. Therefore

$$E_x = \frac{1}{\pi} \int_0^{\infty} |X(\omega)|^2 d\omega$$

- ◆ Consequently, the energy contributed by a real signal by spectral components between  $\omega_1$  and  $\omega_2$  is:

$$\Delta E_x = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |X(\omega)|^2 d\omega$$

## Example

- ◆ Find the energy  $E$  of signal  $x(t) = e^{-at} u(t)$ . Determine the frequency  $W$  (rad/s) so that the energy contributed by the spectral component from 0 to  $W$  is 95% of the total signal energy  $E$ .

- ◆ Take FT of  $x(t)$ :

$$X(\omega) = \frac{1}{j\omega + a}$$

- ◆ By Parseval's theorem:

$$E_x = \frac{1}{\pi} \int_0^{\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi} \int_0^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^{\infty} = \frac{1}{2a}$$

- ◆ Energy in band 0 to  $W$  is 95% of this, therefore:

$$\frac{0.95}{2a} = \frac{1}{\pi} \int_0^W \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi a} \tan^{-1} \frac{\omega}{a} \Big|_0^W = \frac{1}{\pi a} \tan^{-1} \frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1} \frac{W}{a} \implies W = 12.706a \text{ rad/s}$$

- ◆ Note: For this signal, 95% of energy is in the lower frequency band from 0 to  $12.7a$  rad/s or  $2.02a$  Hz!!!

# Three Big Ideas

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1. Extracting a portion of a signal can be modelled by multiplying the signal with a **rectangular window**. However, the sudden changes at the window boundaries modify the signal spectrum.
2. This causes **spectral spreading** to neighbouring frequencies and leakages to higher frequencies. Both can be reduced by using other types of **window functions** such as Hamming or Hanning, which have smooth cut offs.
3. **Discrete Fourier Transform** (DFT) is used to calculate the Fourier Transform in a computer. This is done by taking the windowed portion of the signal **and construct a periodic signal** from it. The result is a sampled Fourier Transform with frequency step  $f_0 = 1/T_0$ , where  $T_0$  is the window function width.